A warm remembrance of Clarence and the days we spent together in Santa Barbara arguing about these very topics. Thank you Georg and team for making this day possible. It’s great we can meet to share and extend Clarence’s incredible contributions to music and research.

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Chords, melodies: a look at harmony by numbers
Using Harmonic Radius

Today I’ll talk about some new research from the past year, focussing on something I call “harmonic radius”. Thanks to my steering group at the Sibelius Academy Helsinki for supporting this research, in particular Tuire Kuusi and Juhani Nuorvala.

https://chordsmelodies.plainsound.org
pdf files of presentation slides and paper(s) for private perusal

For a pre-publication pdf of my paper introducing harmonic radius as well as slides from today's presentation please visit this directory.

Harmony by numbers

- harmony ?

Let’s start with a couple of questions: WHAT do we mean by harmony?

Harmony by numbers

- harmony ?
- numbers ?

and WHY numbers, in what way are they related to harmony?

Harmony by numbers | *"armonia"* = fitting together

- harmony
- numbers ?

James Tenney reminds us of the origin of the word "harmony" as the Greek

Harmony by numbers | *"armonia"* = fitting together

- harmony = relations between pitches other than higher / lower (Tenney)
- numbers ?

and he also offers a simple definition of his own

Harmony by numbers | *“overtones”* = aliquot divisions

- harmony = relations between pitches other than higher / lower
- numbers

and by observing a connection to aliquot divisions of a string

Harmony by numbers | partials

- harmony = relations between pitches other than higher / lower
- numbers = harmonic partial row (Aristotle, Mersenne, Sauveur, Helmholtz, et al.)

the row of harmonic partials (Partialtonreihe) was discovered

Harmony by numbers | rational intervals

- harmony = relations between pitches other than higher / lower
- numbers = harmonic partial row, ratios of partials = intervals (Partch et al.)

Early JI composers like Elsie Hamilton and Harry Partch defined intervals as ratios, including higher partials

Harmony by numbers = relations of harmonic partials

- harmony = relations between pitches other than higher / lower
- numbers = harmonic partial row, ratios of partials

so we can define "harmony by numbers" as ....

Harmony by numbers: measuring harmonicity?

- harmony = relations between pitches other than higher / lower
- numbers = harmonic partial row, ratios of partials
- ratios of smaller numbers are more easily perceived as harmonic; how can "smallness" be measured for any collection of rationally related pitches (chord, scale, mode, gamut...)?

Harmony by numbers: pitch classes?

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Harmony by numbers: primes?

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- what about prime factors and divisibility?

Harmony by numbers: *harmonic radius*

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- numbers = harmonic partial row, ratios of partials
- ratios of smaller numbers are more easily perceived as harmonic; how can "smallness" be measured for any collection of rationally related pitches (chord, scale, mode, gamut...)?
- what about octave-equivalence (notes, pitch classes)?
- what about prime factors and divisibility?
- *harmonic radius* provides a way of comparing the relative harmonicities of pitch sets tuned in rational intonation (JI)

Ordering rational intervals

Since the harmonic partial row presents an infinite gradation of possible rational intervals, how may these intervals be ordered?

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There is no “smallest” or “largest” interval ...

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There is another possibility: proportions of smaller *numbers* make more easily recognisable intervals.

… Benedetti … Tenney … Euler … Barlow

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Since the 1500's, various theorists and composers have sought ways of quantifying how sound combinations manifest various degrees of harmonicity …

Benedetti to Cipriano de Rore

published in "Diversarum speculationum mathematicarum et physicarum liber" (Turin, 1585)

Cum autem ponticulus ita diuiſerit chordam, ut ab uno latere relinquantur duæ quintæ partes, ab alio verò tres quintæ, ex quibus partibus generatur conſonantia diapente; tunc clarè patet, quod eadem proportione tardius erit vnum interuallum tremoris maioris portionis, vno interuallo tremoris minoris portionis, quam maior portio habet ad minorem; hoc eſt tempus maioris interualli ad tempus minoris erit ſeſquialterũ; quare non cõuenient ſimul, niſi perfectis tribus interuallis minoris portionis, & duobus maioris; ita quod eadem proportio erit numeri interuallorum minoris portionis ad interualla maioris, quæ longitudinis maioris portionis ad longitudinem minoris; vnde productum numeri portionis minoris ipſius chordæ in numerum interuallorum motus ipſius portionis, æquale erit producto numeri portionis maioris in numerum interuallorum ipſius maioris portionis; quæ quidem producta ita ſe habebunt, vt in diapaſon, ſit binarius numerus; in diapente verò ſenarius; in diateſſaron duodenarius, in hexachordo maiori quindenarius; in ditono vicenarius, in ſemiditono tricenarius, demum in hexachordo minori quadragenarius: qui quidem numeri non abſque mirabili analogia conueniunt inuicem.

Voluptas autem, quam auditui afferunt conſonantiæ ſit, quia leniuntur ſenſus, quemadmodum cõtra, dolor qui à diſſonantijs oritur, ab aſperitate naſcitur, id quod facilè videre poteris cum conchordantur organorum fiſtulæ.

Giambattista Benedetti, a 16th century mathematician wrote two "letters" to the madrigal composer Cipriano de Rore addressing questions of intonation and harmony

Benedetti to Cipriano de Rore

translation: Thomas Nicholson (2023)

From this, the product of the number of the smaller portion [length] of this string and the number of intervals of the motions of that portion [2·3] will be equal to the product of the number of the larger portion [length] and the number of intervals of that larger portion [3·2]. Indeed, these products will be such that in the octave [1:2] there is the number 2, in the perfect fifth [2:3 the number] 6, in the perfect fourth [3:4 the number] 12, in the major sixth [3:5 the number] 15, in the major third [4:5 the number] 20, in the minor third [5:6 the number] 30, and lastly in the minor sixth [5:8 the number] 40: indeed, these numbers do not agree with each other without a wonderful similarity.

in particular he observed how intervals formed patterns within a common period

Tenney-Benedetti *harmonic distance*

Given a fraction in lowest terms, $\frac{num}{den}$

its harmonic distance is defined as the quantity $\mathrm{HD} = \log_2(num \cdot den)$

– *John Cage and the Theory of Harmony* (1983) by James Tenney

= Lowest Common Partial

Given a fraction in lowest terms, $\frac{num}{den}$

its harmonic distance is defined as the quantity $\mathrm{HD} = \log_2(num \cdot den)$.

If the fraction represents an interval between *two sounding notes*, HD may be interpreted musically as the pitch distance from their *common fundamental* to their *lowest common partial*.

Euler's *Gradus Suavitas*

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Lowest Common Partial (again)

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To calculate the degree of intervals and chords, Euler takes the least common multiple (or lowest common partial) of the pitches, and applies his equation to this number.

Barlow's *indigestibility*

In 1978 composer Clarence Barlow derived his own variation of Euler's formula, scaled to increase the degree of successively higher primes, calling it *indigestibility.* Like Euler, Barlow's formula calculates interval concordance based on the least common multiple.

Barlow’s *specific harmonicity*

In 1978 composer Clarence Barlow derived his own variation of Euler’s formula, scaled to increase the degree of successively higher primes, calling it *indigestibility.* Like Euler, Barlow’s formula calculates interval concordance based on the least common multiple.

For chords and pitch sets, Barlow defines the concept of *harmonicity sum* (the sum of all pairwise harmonicities in a set of pitches) and compares it with the total number of possible pairs to obtain *specific harmonicity.*

Euler´s formula

For a number, expressed (uniquely) as a product of primes:

$$GS\left(\prod_i {p_i}^{k_i}\right) = 1 + \sum_i k_i(p_i - 1)$$

Barlow′s formula

For a number, expressed (uniquely) as a product of primes:

$$\xi\left(\prod_{i} {p_i}^{k_i}\right) = 2\sum_{i} k_i \frac{(p_i - 1)^2}{p_i}$$

The problem

Both Euler's and Barlow's methods for working with chords run into a similar paradox. Consider the major triad (4:5:6) and its intervallic inversion, a minor triad (10:12:15).

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The problem: otonal and utonal chords

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However, the least common multiple of both chords is 60, and both have the same pairwise intervals (3/2, 5/4, 6/5), so neither method numerically represents the difference perceived.

When intervallic inversion is extended to chords including higher primes, the difference between the "otonal" form and its "utonal" (subharmonic) inversion increases.

Chords with same LCM (7-Limit)

Compare higher prime chords with the same LCM and pairwise intervals:

- 7-Limit – 6:7:8:9 and 56:63:72:84

Chords with same LCM (11-Limit)

Compare higher prime chords with the same LCM and pairwise intervals:

- 7-Limit – 6:7:8:9 and 56:63:72:84
- 11-Limit – 8:9:11:12 and 66:72:88:99

Chords with same LCM (13-Limit)

Compare higher prime chords with the same LCM and pairwise intervals:

- 7-Limit – 6:7:8:9 and 56:63:72:84
- 11-Limit – 8:9:11:12 and 66:72:88:99
- 13-Limit – 10:12:13:15 and 52:60:65:78

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It doesn´t make sense to assign the same harmonicity value to each of these pairs.

Measuring harmonicity

To measure the “harmonic distance” of arbitrary sets of tones in a way that *correlates* with harmonicity, it is necessary to express the tones not as fractions, but as natural numbers.

Measuring harmonicity = comparing to harmonic partials

To measure the "harmonic distance" of arbitrary sets of tones in a way that *correlates* with harmonicity, it is necessary to express the tones not as fractions, but as natural numbers: as *partials of a common fundamental,* so that they may be compared against an harmonic partial row.

Measuring harmonicity: generalised harmonic distance

To measure the “harmonic distance” of arbitrary sets of tones in a way that *correlates* with harmonicity, it is necessary to express the tones not as fractions, but as natural numbers: *as partials of a common fundamental,* so that they may be compared against an harmonic partial row.

The simplest way of formulating the “size of numbers” question is to determine “how close”, *on average,* a set of partials lies from its nearest fundamental.

Harmonic Radius measures the partials′ geometric mean

Given a set S of n pitches expressed as partials $\{P°_1, \ldots, P°_n\}$:

$$Harmonic\ Radius\ (S) = \sqrt[n]{\prod_i P_i}$$

Log Radius measures the partials' arithmetic mean

Given a set S of n pitches expressed as partials $\{P°_1, \ldots, P°_n\}$:

$$Harmonic\ Radius\ (S) = \sqrt[n]{\prod_i P_i}$$

$$Log_2\ Radius\ (S) = \log_2 \sqrt[n]{\prod_i P_i} = \frac{1}{n}\sum_i \log_2 P_i\ .$$

Odd partials = pitch classes

Powers of 2 transpose partials by octaves; therefore, *odd* partials represent the first occurrences of each pitch class generated by a given fundamental. Even partials may be reduced to pitch classes by dividing out all powers of 2, obtaining an odd partial.

Odd Radius

Define the odd partial pitch class of a natural number partial $P°$:

$$div2(P)$$

$$Odd\ Radius\ (P) = Harmonic\ Radius\ \left(div2(P)\right).$$

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Odd radius is especially useful in situations where pitches are “octave-equivalent”, recurring in various octaves: e.g., in common scales and gamuts; in chord inversions.

What about divisibility and the size of prime factors?

- compare the two whole tones 7:8 and 8:9

What about divisibility and the size of prime factors?

- compare the two whole tones 7:8 and 8:9; in terms of harmonic radius they are similar: 7:8, which is a more easily tuneable dyad, has a radius value of 7.48, while 8:9 has a slightly larger radius value of 8.49

8:9 is the ratio of two composite numbers

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- on the other hand, the special musical role played by the tertial diatonic tone 8:9 ...

8:9 is the difference between 2:3 and 3:4

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7:8 is the difference between 4:7 and 1:2

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- the septimal tone is a more "distant" construction, tuneable as the difference between the rarely used septimal seventh 4:7 and the octave 1:2

Larger prime factors tend to reduce divisibility

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- the septimal tone is a more "distant" construction, tuneable as the difference between the rarely used septimal seventh 4:7 and the octave 1:2
- larger prime factors (7) introduce more "complexity" by reducing "divisibility"

Prime Limit Radius

Define the greatest prime factor for any natural number:

$$gpf(P), P \geq 2; \;\; gpf(1) = 1$$

$$Prime\ Limit\ Radius\ (P) = Harmonic\ Radius\ (\{P, gpf(P)\}).$$

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Prime Limit Radius captures the difference introduced by higher primes:
7:8 has a prime limit radius value of 6.06 while 8:9 has 4.56.

Comparing two intervals with different divisibility

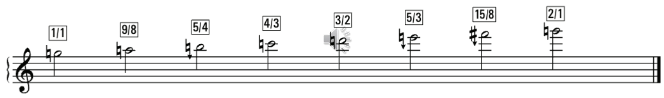
	{15°, 16°} (5-Limit interval; composite)		{3°, 7°} (7-Limit interval; prime numbers)	
harmonic radius = √Benedetti distance	{15°, 16°}	15.49	{3°, 7°}	4.58
prime limit radius	{{15°, 5°}, {16°, 2°}}	6.999	{{3°, 3°}, {7°, 7°}}	4.58
odd radius	{15°, 1°}	3.87	{3°, 7°}	4.58
prime limit odd radius	{{15°, 5°}, {1°, 1°}}	2.94	{{3°, 3°}, {7°, 7°}}	4.58
Euler	LCM{15, 16} = 240	11	LCM{3, 7} = 21	9
Barlow	$\frac{sgn(\xi(16) - \xi(15))}{\xi(15) + \xi(16)}$	-1/13.07	$\frac{sgn(\xi(7) - \xi(3))}{\xi(3) + \xi(7)}$	1/12.95

Working with harmonic radius: an example

- for example, take the major scale, written as a set of fractions:

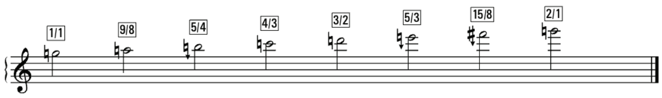
Working with harmonic radius: take a scale of ratios

- for example, take the major scale, written as a set of fractions:
  1/1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, (2/1)



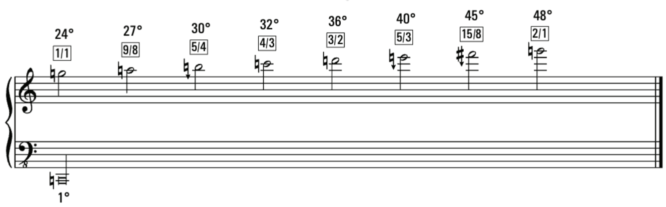
Working with harmonic radius: find the LCM

- for example, take the major scale, written as a set of fractions:
  1/1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, (2/1); the LCM of the denominators is 24



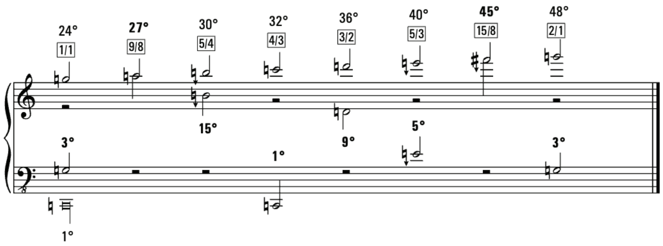
Working with harmonic radius: multiply

- for example, take the major scale, written as a set of fractions:
  1/1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, (2/1); the LCM of the denominators is 24
- multiplying each fraction by 24 obtains the major scale as a set of partials



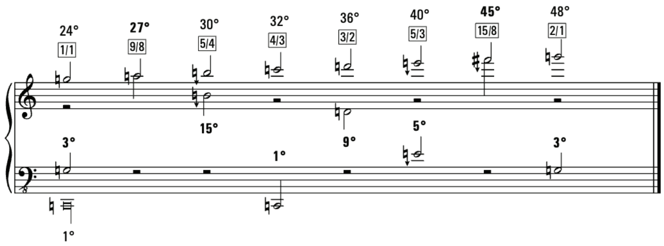
Working with harmonic radius: divide out 2′s

- for example, take the major scale, written as a set of fractions:
  1/1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, (2/1); the LCM of the denominators is 24
- multiplying each fraction by 24 obtains the major scale as a set of partials
- dividing out 2′s reduces each partial to its odd partial pitch class



Working with harmonic radius: fundamental $\neq$ tonic

- note that the fundamental (1°) is the subdominant degree of the scale in this mode (i.e., in the Lydian mode the fundamental is also the tonic); other modes are rotations of the same set of partials



Working with harmonic radius: partials in rising order

- the major scale as a collection of unique odd partial pitch classes: 3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°

Working with harmonic radius: partials are 5-Limit

- the major scale as a collection of unique odd partial pitch classes: 3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- this collection consists of 5-Limit pitch classes (no prime factors > 5)

Working with harmonic radius: higher primes

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- in the 7-Limit, lower odd partials are possible: 15° => 7° and 45° => 21°;

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- in the 7-Limit, lower odd partials are possible: 15° => 7° and 45° => 21°
- in the 11-Limit, 45° or 21° => 11°; in the 13-Limit, 27° => 13°

Working with harmonic radius: higher primes

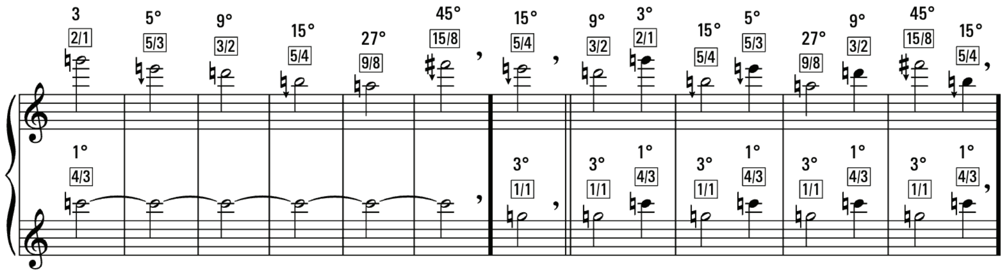
- the major scale as a collection of unique odd partial pitch classes: 3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- this collection consists of 5-Limit pitch classes (no prime factors >5)
- in the 7-Limit, lower odd partials are possible: 15° $=>$ 7° and 45° $=>$ 21°
- in the 11-Limit, 45° or 21° $=>$ 11°; in the 13-Limit, 27° $=>$ 13°, obtaining
- the 7-tone odd partial set with minimum Odd Radius ("acoustic scale") 1°, 3°, 5°, 7°, 9°, 11°, 13°

Working with harmonic radius: subsets (intervals)

- the major scale as a collection of unique odd partial pitch classes:
  3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals)

Working with harmonic radius: subsets (intervals)

- the major scale as a collection of unique odd partial pitch classes: 3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5)



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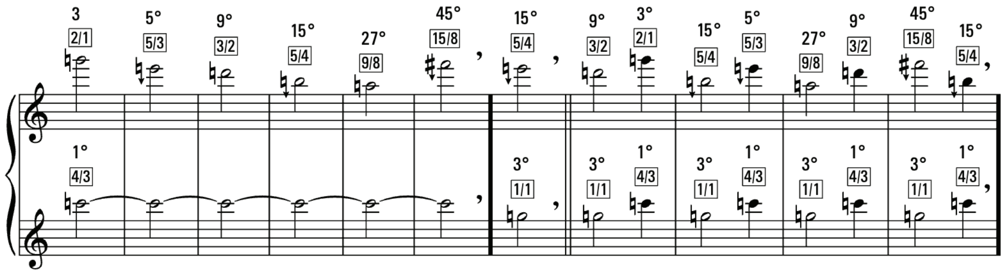
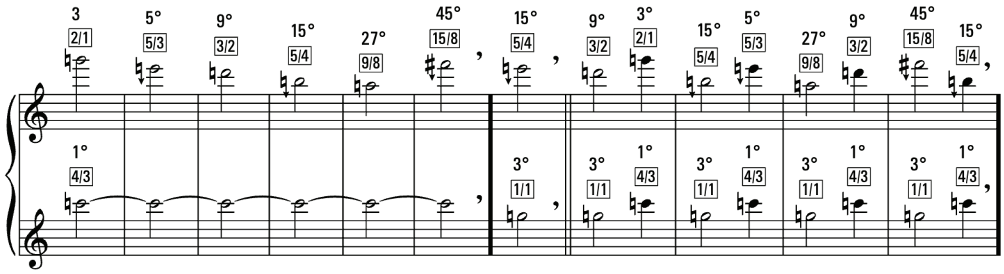
- 1:3 (P5), 1:5 (quintal M3)



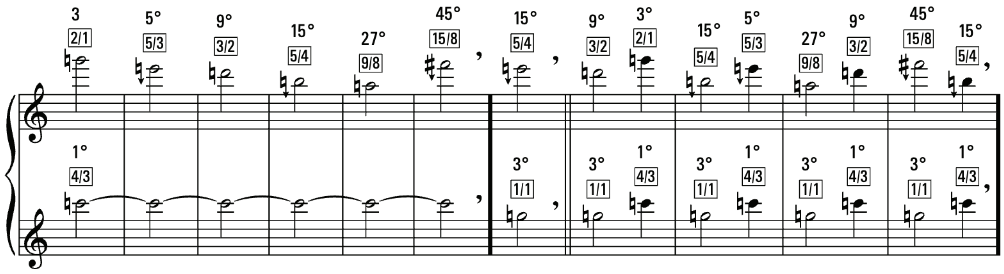
Working with harmonic radius: subsets (intervals)

- the major scale as a collection of unique odd partial pitch classes: 3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5), 1:5 (quintal M3), 1:9 (M2)



Working with harmonic radius: subsets (intervals)

- the major scale as a collection of unique odd partial pitch classes: 3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5), 1:5 (quintal M3), 1:9 (M2), 1:15 (q_m2)



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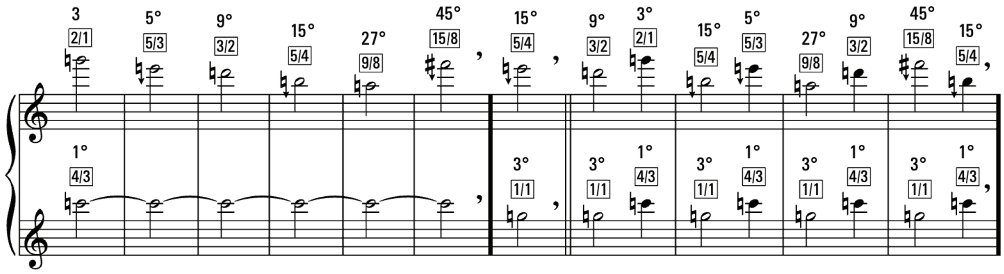

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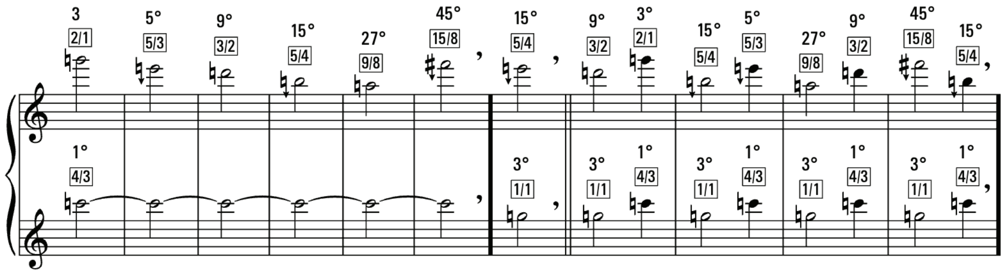
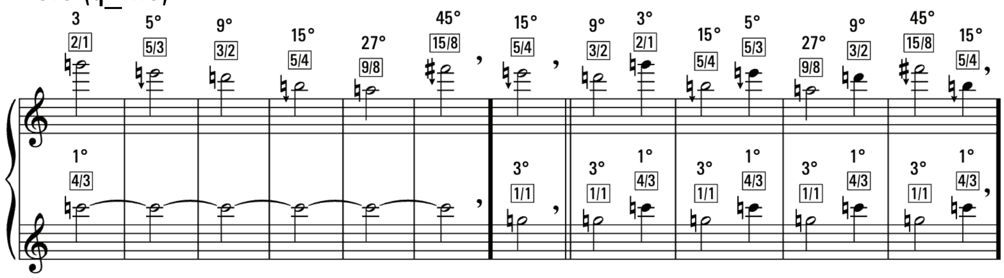
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5), 1:5 (quintal M3), 1:9 (M2), 1:15 (q_m2), 1:27 (tertial m3), 1:45 (q_aug4)



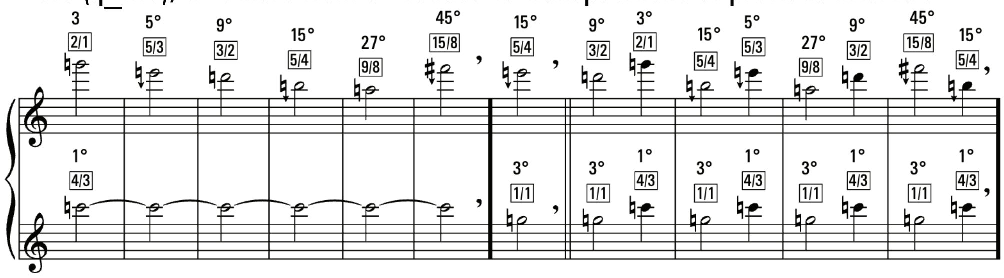
Working with harmonic radius: subsets (intervals)

- the major scale as a collection of unique odd partial pitch classes:
  3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5), 1:5 (quintal M3), 1:9 (M2), 1:15 (q_m2), 1:27 (tertial m3), 1:45 (q_aug4)
  3:5 (q_M6)



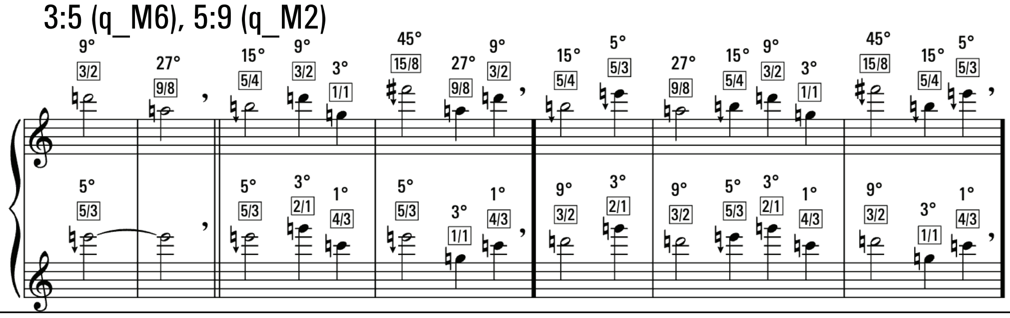
Working with harmonic radius: subsets (intervals) reduce

- the major scale as a collection of unique odd partial pitch classes:
  3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5), 1:5 (quintal M3), 1:9 (M2), 1:15 (q_m2), 1:27 (tertial m3), 1:45 (q_aug4)
  3:5 (q_M6), all others from 3° reduce to transpositions of previous intervals



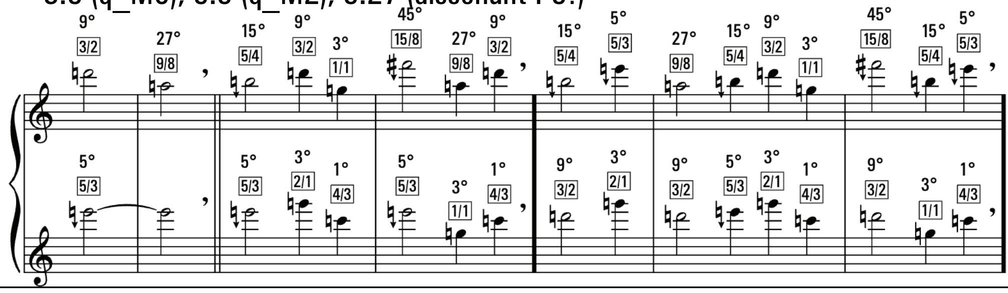
Working with harmonic radius: subsets (intervals)

- the major scale as a collection of unique odd partial pitch classes:
  3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5), 1:5 (quintal M3), 1:9 (M2), 1:15 (q_m2), 1:27 (tertial m3), 1:45 (q_aug4)
  3:5 (q_M6), 5:9 (q_M2)



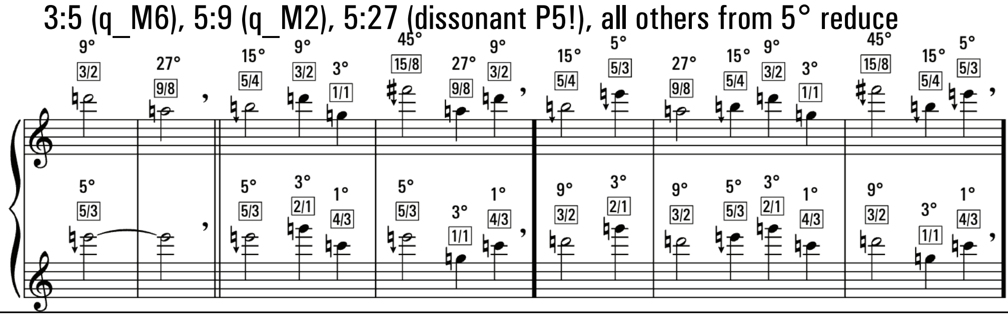
Working with harmonic radius: subsets (intervals)

- the major scale as a collection of unique odd partial pitch classes:
  3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5), 1:5 (quintal M3), 1:9 (M2), 1:15 (q_m2), 1:27 (tertial m3), 1:45 (q_aug4)
  3:5 (q_M6), 5:9 (q_M2), 5:27 (dissonant P5!)



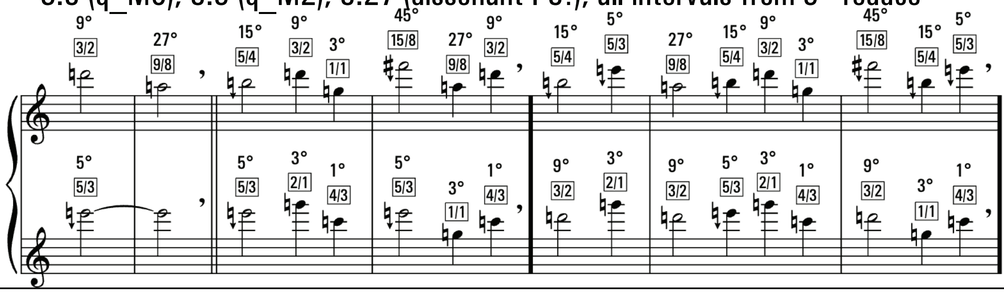
Working with harmonic radius: subsets (intervals) reduce

- the major scale as a collection of unique odd partial pitch classes:
  3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5), 1:5 (quintal M3), 1:9 (M2), 1:15 (q_m2), 1:27 (tertial m3), 1:45 (q_aug4)
  3:5 (q_M6), 5:9 (q_M2), 5:27 (dissonant P5!), all others from 5° reduce



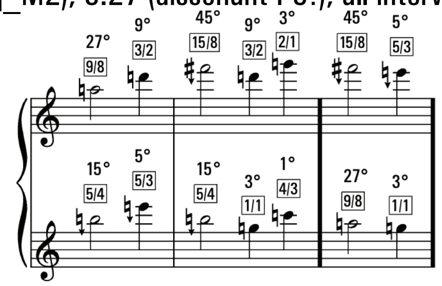
Working with harmonic radius: subsets (intervals) reduce

- the major scale as a collection of unique odd partial pitch classes:
  3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5), 1:5 (quintal M3), 1:9 (M2), 1:15 (q_m2), 1:27 (tertial m3), 1:45 (q_aug4)
  3:5 (q_M6), 5:9 (q_M2), 5:27 (dissonant P5!), all intervals from 9° reduce



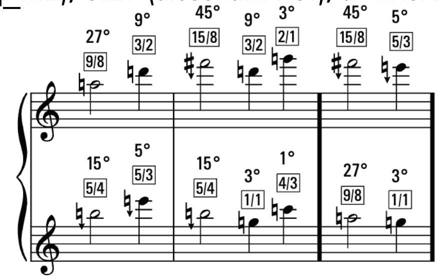
Working with harmonic radius: subsets (intervals) reduce

- the major scale as a collection of unique odd partial pitch classes:
  3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5), 1:5 (quintal M3), 1:9 (M2), 1:15 (q_m2), 1:27 (tertial m3), 1:45 (q_aug4)
  3:5 (q_M6), 5:9 (q_M2), 5:27 (dissonant P5!), all intervals from 15° reduce



Working with harmonic radius: subsets (intervals) reduce

- the major scale as a collection of unique odd partial pitch classes: 3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- 1:3 (P5), 1:5 (quintal M3), 1:9 (M2), 1:15 (q_m2), 1:27 (tertial m3), 1:45 (q_aug4) 3:5 (q_M6), 5:9 (q_M2), 5:27 (dissonant P5!), all intervals from 27° reduce



Working with harmonic radius: subsets (intervals) reduce

- the major scale as a collection of unique odd partial pitch classes: 3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- the most discordant interval is not 27:45, which reduces to 3:5; it is 5:27

Working with harmonic radius: subsets in general

- the major scale as a collection of unique odd partial pitch classes:
  3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- the most discordant interval is not 27:45, which reduces to 3:5; it is 5:27
- a similar approach applies to triads, tetrads, etc.

Working with harmonic radius: averaging

- the major scale as a collection of unique odd partial pitch classes: 3°, 27°, 15°, 1°, 9°, 5°, 45°; in rising order: 1°, 3°, 5°, 9°, 15°, 27°, 45°
- consider all the pairwise subsets (intervals, sounded in closed position)
- the most discordant interval is not 27:45, which reduces to 3:5; it is 5:27
- a similar approach applies to triads, tetrads, etc.
- it is useful to calculate and average values of radius for various textures (chords, melodies)

Working with harmonic radius: chord inversions

- compare the major and minor triads and their inversions:

Working with harmonic radius: major chord

- compare the major and minor triads and their inversions:
- Major = {1°, 3°, 5°}: this is the smallest-radius harmonic triad with three different odd partial pitch classes; closed position inversions, in order of radius: {2°, 3°, 5°}, {3°, 5°, 8°}, {5°, 8°, 12°}

Working with harmonic radius: minor chord

- compare the major and minor triads and their inversions:
- Major = {1°, 3°, 5°}: this is the smallest-radius harmonic triad with three different odd partial pitch classes; closed position inversions, in order of radius: {2°, 3°, 5°}, {3°, 5°, 8°}, {5°, 8°, 12°}
- Minor = {3°, 5°, 15°} with its inversions; clearly all bear larger radius values: {6°, 10°, 15°}, {10°, 12°, 15°}, {15°, 20°, 24°}

Working with harmonic radius: melodies, intervals

- compare the major and minor triads and their inversions:
- Major = {1°, 3°, 5°}: this is the smallest-radius harmonic triad with three different odd partial pitch classes; closed position inversions, in order of radius: {2°, 3°, 5°}, {3°, 5°, 8°}, {5°, 8°, 12°}
- Minor = {3°, 5°, 15°} with its inversions; clearly all bear larger radius values: {6°, 10°, 15°}, {10°, 12°, 15°}, {15°, 20°, 24°}
- each triad has 3 intervals (pairs of partials); when these are reduced to lowest terms, it is clear that, taken *melodically,* the two triads are equivalent:
  {4°, 5°, 6°} => {{4°, 5°}, {2°, 3°}, {5°, 6°}}
  {10°, 12°, 15°} => {{5°, 6°}, {2°, 3°}, {4°, 5°}}

Comparing triads using various forms of radius

	major triad {4°, 5°, 6°}		minor triad {10°, 12°, 15°}	
harmonic radius	{4°, 5°, 6°}	4.93	{10°, 12°, 15°}	12.16
prime limit radius	{{4°, 2°}, {5°, 5°}, {6°, 3°}}	3.91	{{10°, 5°}, {12°, 3°}, {15°, 5°}}	7.16
odd radius	{1°, 5°, 3°}	2.47	{5°, 3°, 15°}	6.08
prime limit odd radius	{{1°, 1°}, {5°, 5°}, {3°, 3°}}	2.47	{{5°, 5°}, {3°, 3°}, {15°, 5°}}	5.06
Tenney intersection	28 coincident partials / 60	1/2.4	12 coincident partials / 60	1/5
Euler	LCM{4, 5, 6} = 60	9	LCM{10, 12, 15} = 60	9
Barlow and Benedetti only provide a pairwise sum of intervals, obtaining the same result for both chords.				

Working with harmonic radius: some first conclusions

- only by evaluating the harmonic radius of an entire set as partials of a common fundamental, as well as considering various relevant subsets, is it possible to quantitatively determine differences in overall harmonicity

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Working with harmonic radius: some first conclusions

- only by evaluating the harmonic radius of an entire set as partials of a common fundamental, as well as considering various relevant subsets, is it possible to quantitatively determine differences in overall harmonicity
- a microtonal musical gamut often collects many more intonational variants than are active at a given time; it is crucial to determine which subsets are relevant *in a given musical texture and context* and to calculate average radius across those subsets

Tolerance and Intonation

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- results may be filtered by prime limit radius or by limiting prime factors of partials included in the search set
- for a given note in a scale *(svara)*, varying *śruti* may be contextually determined
- for example, in the chromatic raga *Todi*, the optimal 19-limit tuning of the entire scale (7 notes) varies from the optimal tunings of individual notes against the drone

Harmony by numbers

By defining a measure of relative harmonicity for different set sizes, *harmonic radius* offers a novel means for systematic analysis of pitch sets, harmonicity, and intonation, which may be easily estimated by composers and performers in real-time.

Harmony by numbers: hexatone

A current project-in-progress is an open source touch-compatible isomorphic layout web app. It allows playback (using built-in or external sounds) of any microtonal pitch-sets, and is compatible with an ordinary MIDI keyboard, Lumatone, LinnStrument, MTS-ESP, and various MTS and MPE softsynths:

hexatone.plainsound.org

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The app has a number of presets including various prime-limited odd partial pitch class sets for experimentation and testing.

Harmony by numbers: scale workshop

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Pitches in the hexatone can be customised using the commonly used scala file format, developed by Manuel Op de Coul. Scala files can be edited by hand or generated algorithmically using the open source project Scale Workshop, developed by Sevish (Sean Archibald) and Lumi Pakkanen:

scaleworkshop.plainsound.org

https://chordsmelodies.plainsound.org
pdf files of presentation slides and paper(s) for private perusal

Marc Sabat
www.plainsound.org
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Chords, melodies: a look at harmony by numbers _Using Harmonic Radius_